

Leon Simon '83

Xiu's sem. $\Sigma\mu$. '15/

Asymptotics for a class of nonlinear evolution eqs.

"Łojasiewicz-Simon technique."

1st half only.

Σ - smooth cpt manifold.

$E(u) = \int_{\Sigma} E(x, u, \nabla u)$ - energy functional

$E(x, z, p)$ strictly convex in p , analytic in z, p (uniformly for $|z| + |p| \leq B$).

$M(u) =$ - gradient of $E(u)$. $\langle M(u), v \rangle = - \frac{d}{ds} E(u + s \cdot v) \Big|_{s=0}$.

Smooth nonlin. elliptic operator (Friedrichs diff. / or $C^2(\Sigma)$).

Assume $M(0) = 0$ - crit. pt.

$(a^{ij}(x)) \geq \epsilon \cdot I$.

Linearization: $Lu = DM(0)(u) = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} a^{ij} \partial_j u) + \text{l.o.t.}$

Evolution: $u_t - M(u) = f(x, t) \rightarrow$ exp. ly. decaying.

Norms: $\|u\| = \|u\|_{L^2(\Sigma)}$

$\|u\|_h = \|u\|_{W^{h,2}(\Sigma)}$

$|u|_{h,\mu} = |u|_{C^{h,\mu}(\Sigma)}$.

Choose h large enough (dim) s.t. $W^{h-1,2} \hookrightarrow C^2$.

Theorem: Given $\varepsilon > 0$, $\exists \delta > 0$ and $\lambda \in (0, 1/2)$ s.t.

assuming $\|f(t)\|_h + \|\dot{f}(t)\|_h + \|\ddot{f}\| \leq \delta e^{-\lambda t}$

and given any φ with $\|\varphi\|_{h,2} \leq \delta$, there exists a

solution $u(x, t)$ with $u(x, 0) = \varphi(x)$ and $T_x > 0$ s.t.

- $T_x < \infty$ and $\sup_{[0, T_x)} \|u(t)\|_h \leq \delta$ and either
- $T_x = \infty$, $E(u) \rightarrow E(0)$, and $u \xrightarrow{C^2} w$ with $M(w) = 0$.

n.b. $w \neq 0$ a.p.!

Corollary: Subseq. conv $\Rightarrow$ smooth convergence (here).

$$u(t_k) \rightarrow w \text{ for some } t_k \rightarrow \infty$$

$$\Rightarrow \quad \quad \quad \text{for all } t_k \rightarrow \infty.$$

2nd half of paper: Uniqueness of ^{smooth} tangent cones at isolated singularities of min Γ submanifolds

$$t = t_{\min}: \quad \ddot{u} - \dot{u} + M(u) + R(u) = f(x, t)$$



Proof of Theorem:

- ① Łojasiewicz-Simon inequality
- ② Stability Lemma
- ③ 3-interval method.

① classical Łojasiewicz Theorem: ¹⁹⁶⁵ $f(x)$ analytic fun on $D \subset \mathbb{R}^n$, with $f(x_0) = a$, $\nabla f(x_0) = 0$.

Let $d(x) = \text{dist}(x, \{ \nabla f = 0 \})$. $\exists D' \ni x_0$, $\theta \in (0, 1/2]$, $\gamma \geq 2$, s.t.

(a) $|f(x) - a| \geq d(x)^\gamma$

(b) $|\nabla f(x)| \geq c |f(x) - a|^{1-\theta} \quad \forall x \in D'.$

Idea of proof:

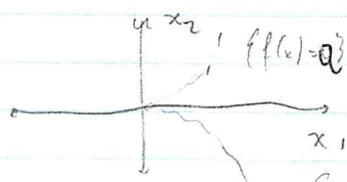
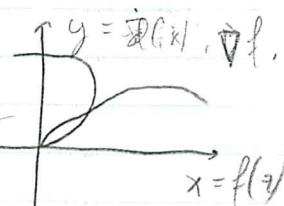


Image of $\mathbb{R}^n$



Semi-analytic set not unit. x-axis: Newton-Puiseux: $y \geq x^{1/2}$

$$-f \geq f^{1-\theta}$$

3

Con: $\{\nabla f = 0\}$ is a def. retract of a ubld.

False for C^∞ func.



Can do:

$$|\Delta u| \leq \int |u| \leq \sqrt{T} \left(\int |u|^2 dt \right)^{1/2} \leq \sqrt{T} \Delta E.$$

Theorem: (Log-Siman inequality) $\exists \theta \in (0, 1/2], \forall \beta \geq 2,$

$0 < \sigma < \beta$ (arbitrary) s.t.

$$\|u\|_{2,\mu} < \sigma$$

$$\Rightarrow (a) \quad \|M(u)\| \geq \left(\inf_{w \in S} \|u - w\| \right)^\theta$$

$$S = \{w \in C^2(\Sigma) : \|w\|_2 < \beta, m(w) = 0\}.$$

$$(b) \quad \|M(u)\| \geq C |\varepsilon(u) - \varepsilon(0)|^{1-\theta}$$

Proof: $L = DM(0) : C^{2,\mu}(\Sigma) \rightarrow C^{0,\mu}(\Sigma)$

$K_0 = \ker L$, fin. d. [bigger space.]

$$K_\pm = (\ker L)^\perp$$

$L : K_\pm \cap C^{2,\mu} \rightarrow K_\pm \cap C^{0,\mu}$ invertible (Schauder).

implicit fun thm: $\exists \varepsilon, \delta$ s.t.

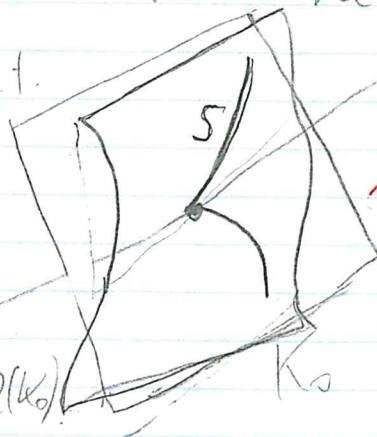
$\forall u_0 \in K_0$ w/ $\|u_0\|_{2,\mu} < \varepsilon$

$\exists! l(u_0) \in K_\pm$ w/ $\|u_0\|_{2,\mu} < \delta$ s.t.

$M(u_0 + l(u_0)) \in K_0$
i.e. $\pi_\pm \circ M \neq 0$. $l(K_0)$

$m(u) = 0 \in K_0 \Rightarrow u = \pi u + l(\pi u)$

$\therefore S = \{m(u) = 0\} \subset l(K_0)$



~~$K_\pm$ (s-d.)~~
Should write K_0 !

Define $f(u_0) = \mathcal{E}(u_0 + l(u_0)) - \mathcal{E}(0)$.

Claim: $(\nabla f(u_0), \underbrace{u_0}_{\in K_0}) = (\underbrace{m(u_0 + l(u_0))}_{\in K_0}, \underbrace{u_0 + D l(u_0)u}_{\in K_I})$

SINCE this is on K_0 , have

$\nabla f(u_0) = m(u_0 + l(u_0)) = (m(u_0, l(u_0)), u_0)$.

($\therefore m(u_0) = \{u_0 + l(u_0) \mid \nabla f(u_0) = 0\}$)

f captures all interesting info.

Claim: f is real-analytic!

H. Complexify $\mathcal{E}(x, z, p)$, hol. in z, p .

$\Rightarrow \mathcal{E}(u) : C^{2,p}_b \rightarrow C^{0,p}_b$

C -differentiable (Fréchet.)

d. f.t. : l^p also C -diff, also $f^C(u_0) = \mathcal{E}^C(u_0 + l^C(u_0)) : K_0^C \rightarrow \mathbb{C}$.

Real part is real-analytic. $\square$

Proof of inequality:

LHS = $m(u) = m(u_0 + l(u_0) + \underbrace{u}_{\in K_0})$
 (obv) $= \nabla f(u_0) + m(u_0 + l(u_0) + u) - m(u_0 + l(u_0))$

$= \nabla f(u_0) + \int_0^1 (Dm(u_0 + l(u_0) + su), u) ds$

$= \nabla f(u_0) + L_{u_0} u + L_1 u$

du: $u \in K_0$.

depn of u !

$u \in K_0$

where $L_1 = \int_0^1 (DM(u_0 + l(u_0) + s u_\pm) - DM(0)) ds$

M smooth, $|u_0 + l(u_0) + s u_\pm|_{2\mu} < \varepsilon \Rightarrow \|L_1\|_{L^2} < C \cdot \varepsilon$

$\Rightarrow \|u(u)\| \geq \|\nabla f(u_0) + \frac{L}{n} u_\pm\| - \|L_1 u_\pm\|$

$\nabla f(u_0) \perp A_\pm$

$\geq \frac{1}{\sqrt{2}} (\|\nabla f(u_0)\| + \|L u_\pm\|) - C \varepsilon \cdot \|u_\pm\|_{H^2}$

elipt = el.?

$\geq \frac{1}{\sqrt{2}} \|\nabla f(u_0)\| + C(1-\varepsilon) \|u_\pm\|_{H^2}$

$\approx \varepsilon(u)$

RHS: $\varepsilon(u) - \varepsilon(0) = f(u_0) + \varepsilon(u_0 + l(u_0) + u_\pm) - \varepsilon(u_0 + l(u_0))$

$= f(u_0) + \int_0^1 (u(u_0 + l(u_0) + s u_\pm), u_\pm) ds$

ALREADY
DRAW
= MY IDEAS

of absorbing the difference in energy into smaller of $\delta^{\frac{1}{2}}$.

$= f(u_0) + (u(u_0 + l(u_0)), u_\pm) \rightarrow 0$

$+ \int_0^1 \int_0^1 (DM(u_0 + l(u_0) + s \cdot t \cdot u_\pm / s u_\pm), u_\pm) dt ds$

$= f(u_0) + \frac{1}{2} (L u_\pm, u_\pm) + (L_2 u_\pm, u_\pm)$

where

$L_2 = \int_0^1 \int_0^1 s (DM(u_0 + l(u_0) + s \cdot t \cdot u_0) - L) dt ds$

as before

$\|L_2\| < C \varepsilon$

$\Rightarrow |\varepsilon(u) - \varepsilon(0)| \leq |f(u_0)| + C \|u_\pm\|_{H^2}^2$

better even.

Recall: $|f(u)|^{1-\theta} \leq C |\nabla f(u)|$, $\frac{1}{2} \leq 1-\theta \leq 1$

$$\Rightarrow \text{Hölder} \quad |\varepsilon(u) - \varepsilon(0)|^{1-\theta} \leq |f(u)|^{1-\theta} + C \|u_{\pm}\|_{H^2}^{2-2\theta}$$

$$\begin{aligned} (1/\|u_{\pm}\|_2 \leq C(\varepsilon) \|u_{\pm}\|_2 \leq \varepsilon) &\Rightarrow \leq C(|\nabla f(u)| + \varepsilon^{1-2\theta} \|u_{\pm}\|_{H^2}) \\ &\leq C \|u(u)\| \quad \square \end{aligned}$$

Remark: If S is smooth at 0, i.e. regular at pt., then $\nabla f \equiv 0$. But no way of knowing this a.p.!

$\Rightarrow \theta = 1/2$ for exp'l convergence

(2) Stability lemma: $\dot{u} = u(u) = f(x, t)$ as above

Doesn't assume $\|u(t)\|_{2,p} < \sigma$ no significance yet!

Use exponential decay. $\varepsilon(u(t)) > \varepsilon(0) - \delta$ for $t \in [t_1, t_2]$

and $\|f(t)\| \leq (1-\alpha) \|\dot{u}(t)\|$ for some $\alpha \in (0, 1)$

Then $\sup_{t \in [t_1, t_2]} \|u(t) - u(t_1)\| \leq \int_{t_1}^{t_2} \|\dot{u}(t)\| dt \leq (2\theta)^{-1} \left(|\varepsilon(u(t_2)) - \varepsilon(0)|^{1-\theta} + \delta^\theta \right)$

Proof: $\|\dot{u}\|^2 + \|u(u)\|^2 - 2(u, u(u))$

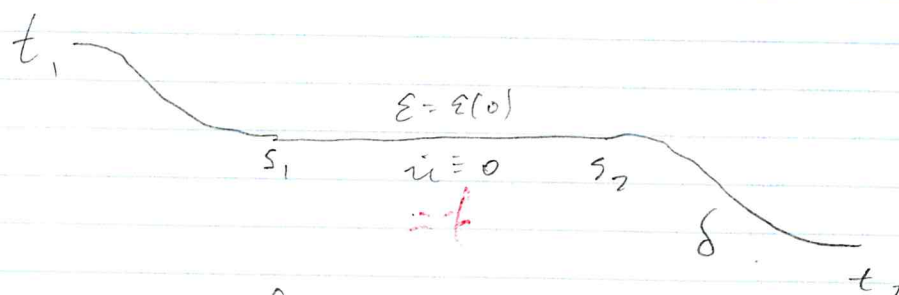
$$= \|\dot{u}\|^2 \leq (1-\alpha)^2 \|\dot{u}\|^2 \leq (1-\alpha) \|\dot{u}\|^2$$

$$\Rightarrow (u, u(u)) \geq \alpha/2 \|\dot{u}\|^2 + \alpha/2 \|u(u)\|^2$$

$$\geq \alpha \|\dot{u}\| \|u(u)\|$$

$$\Rightarrow -\frac{d}{dt} E(u(t)) = (u(t), u'(t)) \geq \alpha \|u'(t)\|$$

$\Rightarrow E$ dec.



$t \leq s_1$:

$$-\frac{d}{dt} (E(u(t)) - E(s_1))^\theta \geq \theta (E(u(t)) - E(s_1))^{\theta-1} \alpha \|u'(t)\|$$

$$t \geq s_2: \quad \geq \alpha \theta \|u'(t)\|$$

$$\int_{t_1}^{s_1} \|u'(t)\| dt \leq (\alpha \theta)^{-1} (E(u(t_1)) - E(s_1))^\theta$$

$$\int_{s_1}^{s_2} \|u'(t)\| dt = 0$$

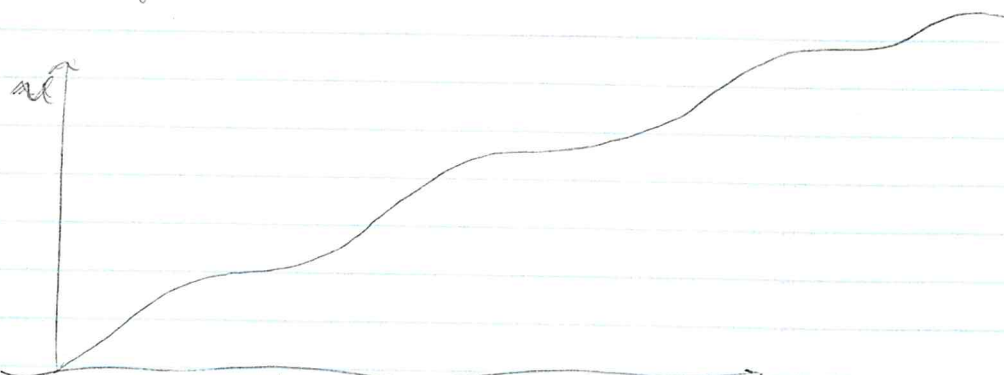
$$\int_{s_2}^{t_2} \|u'(t)\| dt \leq (\alpha \theta)^{-1} \delta^\theta \quad \square$$

Point of lemma: $L^1 \not\subset L^2(E \times [0, \infty))$!

Now, have either $\|u'\| \leq \delta e^{-\alpha t}$

$$\text{or} \quad \int_{t_1}^{t_2} \|u'\| \leq (\Delta E)^\theta$$

Problem?



Seminar: $\exists T$, s.t. $\|u'\| \leq \delta e^{-\alpha t}$ for $t < T$,
 $\geq \alpha \theta e^{-\alpha t}$ for $t \geq T$.

Std parabolic ests: $k \geq 0$:

$$\sup_{[t_1, t_2]} (\|u(t)\|_k + \|\dot{u}(t)\|_k) + \left(\int_{t_1}^{t_2} \|\dot{u}(t)\|_{k+1}^2 + \|u(t)\|_{k+1}^2 dt \right)$$

$$\leq C \left(\|u(t_1)\|_k + \|\dot{u}(t_1)\|_k + \left(\int_{t_1}^{t_2} (\|\dot{f}(t)\|_{k-1}^2 + \|f(t)\|_{k-1}^2 + \|\dot{u}(t)\|_{k-1}^2) dt \right) \right)$$

We can assume m and $T_x > 0$ are s.t.

$$\|u(t)\|_k + \|\dot{u}(t)\|_k < \delta, \quad \text{on } [0, T_x)$$

$$= \delta, \quad \text{at } t = T_x$$

Then $w = u$ satisfies

$$w - Lw - Pw = g (=f)$$

where $P, g \rightarrow 0$ as $\delta \rightarrow 0$.

③ 3-interval chain rule

let $0 < \varepsilon < \min\{\lambda, (L)/\lambda; > 0\}$.

~~then~~ $\exists \delta_0 > 0$ s.t. if $\delta_1 < \delta_0$ with $\|g(t)\| + \|\dot{g}(t)\| < \delta_0$

$$\text{then } \sup_{[T+1, T+2]} e^{\varepsilon t} \|w(t)\| \geq \max \left\{ \sup_{[T, T+1]} e^{\varepsilon t} \|w(t)\|, \delta_1^{1/2} \right\}$$

$$\Rightarrow \sup_{[T+1, T+3]} e^{\varepsilon t} \|w(t)\| \geq \sup_{[T+1, T+2]} e^{\varepsilon t} \|w(t)\|$$

"if you grow you keep growing..."

$$u(t) = e^{-\lambda t} u(0) + \int_0^t e^{-\lambda(t-s)} f(s) ds$$

Proof by contrad.: Assume claim fails for $\delta_k \rightarrow 0$,
and put $w_k = \tilde{w}_k(t - T_k)$ with $\sup_{[1,2]} \|\tilde{w}_k(t)\| = 1$.

then $1 \geq \sup_{[0,1]} e^{\lambda t} \|w_k(t)\|_{L^2} \Rightarrow w_k \xrightarrow{W^{1,2}([0,3])} u$

but $1 \geq \sup_{[2,3]} e^{\lambda t} \|w_k(t)\|_{L^2}$ any $\epsilon > 0$

$$\delta_k \rightarrow 0 \Rightarrow P_k g_k \rightarrow 0$$

$$\Rightarrow \dot{w} - \lambda w = 0$$

$$\Rightarrow w = \sum a_j e^{-\lambda_j t}, \quad \{a_j\} \in \mathbb{R}$$

$$e^{\lambda t} \|w\|^2 = \sum a_j^2 e^{2(\lambda - \lambda_j)t}$$

is strictly convex!

this is now being determined.

End proof of claim: Let $\delta \in \delta_k \setminus 0$.

Lemma $\Rightarrow$ if $\|u(T_1)\| \geq \delta^{1/2} e^{-\lambda T_1}$

$$\text{Then } \|u(t)\| \geq c \delta^{1/2} e^{-\lambda t} \quad \forall T_1 \leq t \leq T_2 - 1.$$

Else: if $\|u(T_1)\| < \delta^{1/2} e^{-\lambda T_1}$ for some $T_1 < T_2$, replace T_1 by T_1 .

$$0 < t < T_1: \|u(t)\| - \|u(0)\| \leq \int_0^t \|u\| dt \leq \int_0^t \delta^{1/2} e^{-\lambda s} ds$$

$$T_1 < t < T_2 - 1: c \delta^{1/2} \leq \frac{1}{2}, \text{ apply Lemma. } \leq c \delta^{1/2}$$

Step

$$\begin{aligned}
 \sup_{[t_r, T_r-1]} \|u(t)\| - \|u(T_r)\| &\leq \int_{T_r}^{T_r-1} \|\dot{u}(t)\| dt \\
 &\leq C(\delta^\theta + |\varepsilon(u(T_r)) - \varepsilon(0)|^\theta) \\
 &\leq C(\delta^\theta + \|u(T_r)\|_1^\theta).
 \end{aligned}$$

$$\left(\text{Start} \Rightarrow \|u(T_r)\|_1 \leq C\delta^{1/2} \right)$$

$$\sup_{[0, T_r-1]} \|u(t)\| \leq C\delta^{1/2} \leq C\delta_1^{1/2}$$

$$\delta_1 \text{ suff. small} \Rightarrow \sup_{[0, T_r-1]} (\|u(t)\|_e + \|\dot{u}(t)\|_e) \leq C\delta_1^{1/2}$$

$$\text{if } T_r < \infty \Rightarrow T_r = \infty.$$

□